THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Problem Set 1

1. Suppose that $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$ and $\mathbf{v} = -6\mathbf{i} - 8\mathbf{j}$. Find

   (a) $\mathbf{u} \cdot \mathbf{v}$,

   (b) $|\mathbf{u}|$ and $|\mathbf{v}|$,

   (c) the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. Let $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

   (a) Express $\mathbf{a} = \mathbf{u} + \mathbf{v}$ such that $\mathbf{u}$ is parallel to $\mathbf{b}$ and $\mathbf{v}$ is orthogonal to $\mathbf{b}$.

   (b) Find the area of parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

3. Let $A = (3, 3, 0)$, $B = (-2, -3, 2)$ and $C = (1, 0, 3)$ be three points in $\mathbb{R}^3$. Find the volume of the tetrahedron $OABC$.

4. Let A and B be two points in $\mathbb{R}^n$ and let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

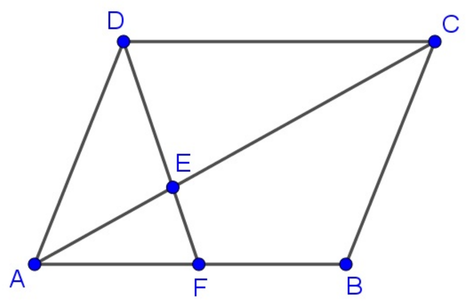
   Suppose that C is a point on AB such that $AC : CB = r : s$, where $r, s \in \mathbb{R}$. Show that

$$\overrightarrow{OC} = \frac{1}{r+s}(r\mathbf{b} + s\mathbf{a}).$$

5. Let $\mathbf{p}$ and $\mathbf{q}$ be nonzero vectors in $\mathbb{R}^n$ such that they are not parallel and let $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

   Prove that if $a_1\mathbf{p} + a_2\mathbf{q} = b_1\mathbf{p} + b_2\mathbf{q}$, then $a_1 = b_1$ and $a_2 = b_2$.

6.



   In the above diagram, $ABCD$ is a parallelogram and F is a point on AB.

   Suppose that DF and AC intersect at the point E such that $DE : EF = \lambda : 1$, where $\lambda > 0$.

   Let $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AD} = \mathbf{q}$, $\overrightarrow{AE} = h\overrightarrow{AC}$ and $\overrightarrow{AF} = k\overrightarrow{AB}$, where $h, k > 0$.

   (a) i. Express $\overrightarrow{AE}$ in terms of h, $\mathbf{p}$ and $\mathbf{q}$.

   ii. Express $\overrightarrow{AE}$ in terms of λ, k $\mathbf{p}$ and $\mathbf{q}$.
   Hence, show that $\lambda = \dfrac{1}{k}$.

   (b) Given that $|\mathbf{p}| = 3$, $|\mathbf{q}| = 2$ and $\angle DAB = \dfrac{\pi}{3}$.

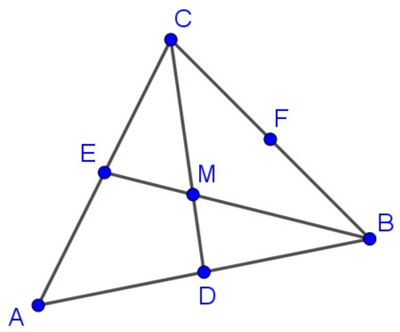
   i. Find $\mathbf{p} \cdot \mathbf{q}$.

   ii. Suppose that DF is perpendicular to AC.

   (1) Express $\overrightarrow{DF}$ in terms of k, $\mathbf{p}$ and $\mathbf{q}$, and so find the value of k.

(2) Using (a), find the length of AE.

7.



In the above diagram, A, B, C are three distinct points in $\mathbb{R}^2$ and let $\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AC} = \mathbf{q}$.

Suppose that D, E and F are mid-points of AB, AC and BC respectively, M is the intersection of CD and BE.

(a) Suppose that $CM : MD = r : 1$ and $BM : ME = s : 1$, where $r, s > 0$.

i. Express $\overrightarrow{AM}$ in terms of r, $\mathbf{p}$ and $\mathbf{q}$.

ii. Express $\overrightarrow{AM}$ in terms of s, $\mathbf{p}$ and $\mathbf{q}$.

iii. Hence, show that $r = s = 2$ and $\overrightarrow{AM} = \dfrac{1}{3}(\mathbf{p} + \mathbf{q})$.

(b) Prove that three medians AF, BE and CD of $\triangle ABC$ intersect at the point M.

Also, prove that $CM : MD = BM : ME = AM : MF = 2 : 1$.

8. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ and let

$$p(t) = \sum_{i=1}^{n} (a_i - b_i t)^2$$

be a polynomial.

By using the fact that $p(t) \geq 0$ for all $t \in \mathbb{R}$, prove that the Cauchy Schwarz inequality holds, i.e.

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right)$$

and the equality holds if and only if $a_1 = tb_1, a_2 = tb_2, \ldots, a_n = tb_n$ for some $t \in \mathbb{R}$.